Elastoviscoplastic Buckling Behavior of Simply Supported Columns

Y. Song*
Georgia Institute of Technology, Atlanta, Georgia 30332
and
G. J. Simitses†
University of Cincinnati, Cincinnati, Ohio 45221

The elastoviscoplastic buckling behavior of simply supported columns is investigated. The analysis is based on an incremental formulation that can deal with nonlinear geometrical and constitutive relations. Both viscoelastic and viscoplastic models are discussed, and Bodner-Partom's constitutive equations are used in the viscoplastic case. A numerical example is given to demonstrate the buckling behavior of columns under various temperature and loading levels.

Introduction

THE problem of stability of columns is one of the fundamental problems in mechanics. The earliest work can be traced to Euler, who first introduced the elastic critical load for slender columns that is still widely used in engineering. In the last four decades, considerable attention has been focused on the buckling behavior of viscoelastic materials. The reason for continuing research is that current engineering requires not only the establishment of critical loads but also the details of the response of the structure under complex loading conditions, which include elevated temperature, time-dependent behavior, etc. In the high-temperature case, life prediction is of primary importance.

With regard to solution methodology, Stubstad and Simitses[1] proposed a differential formulation for one-dimensional problems that involves nonlinear kinematic and nonlinear material effects. The method can decouple the temporal and spatial dependence so that the general solution can be treated as a sequential combination of solutions of a nonlinear boundary value problem and a nonlinear initial value problem. The two groups can then be solved separately. Using an integral transform method, Dost and Glockner[2] investigated the dynamic stability of viscoelastic perfect columns. Subsequently, Szyszkowski and Glockner[3] studied the dynamic response of imperfect viscoelastic columns with Laplace transforms being used in both references.[2,3] Because of the complexity of the inverse of the transform, some numerical calculations were employed in order to obtain the final results. Vinogradov[4] suggested a quasielastic method to study the behavior of viscoelastic beam columns and gave some comparisons with analytical results. In order to solve the viscoelastic problem efficiently, Stubstad and Simitses[5] introduced a bounding solution scheme for geometrically nonlinear viscoelastic problems. Both upper and lower bounds were established by bounding the convolution integral of the governing nonlinear Volterra-type integral equation. This method can substantially reduce the computational effort required for numerical evaluation.

The models used by most researchers correspond to viscoelastic behavior with limited or unlimited creep. Figure 1 shows the mechanisms of two types of material models. The shortcoming of these kinds of material models is that they assume that creep will occur at any stress level. However, as proposed by Perzyna[6] and confirmed by other investigators, creep is physically meaningful only after the stress reaches some level, which implies that a threshold is needed in the constitutive equation. It should be realized that since the small deflection introduced by the deviation from the true material behavior can lead to further loading and further deflections, the use of these viscoelastic material models needs special care as it implies certain limitations. Another limitation for these material models is that they can only be applied to rather long columns.

The unified constitutive equations developed in recent years, on the other hand, can serve as the material model in the analysis of buckling of columns. The model can cover elastic, inelastic material behavior, and temperature effects, simultaneously. Unlike the traditional approach, it does not separate the inelastic deformation into time-dependent (creep) and time-independent (plastic) deformations. Instead, it describes the deformation in terms of not only current mechanical quantities but also in terms of several state variables. The state variables are employed to characterize the deformation history. Thus, the path-dependent behavior of material can be better represented.

There have been several unified constitutive models advanced,[7] and among them the Bodner-Partom's model[8] has application over a wide range of strain rates and temperature. The equation is expressed in exponential form, which has the desirable feature that the strain rate is small, essentially negligible, at low stress levels until a threshold value is reached and it then increases with a slope determined by a parameter. Both scalar and tensor internal variables are introduced into the constitutive equation to represent isotropic hardening and directional hardening. The model appears to be in good agreement with experimental tests, as noted in Refs. 8 and 9.

The difficulty in using unified constitutive equations is the increase in mathematical complexity. The constitutive equations are a group of strongly coupled nonlinear differential

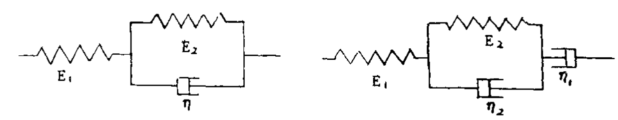

Fig. 1 Three- and four-element viscoelastic models.

Received June 26, 1990; revision received Oct. 18, 1990; accepted for publication Oct. 24, 1990.

*Research Assistant, Aerospace Engineering.

†Professor and Head Department of Aerospace Engineering and Engineering Mechanics. Associate Fellow AIAA.

equations. In general, numerical methods are employed to solve a specific boundary value problem or initial value problem. In order to avoid the large number of subiterations in the calculations of a finite element procedure, Tanaka and Miller[10] proposed a unified numerical method to integrate the stiff time-dependent constitutive equations. The asset of the method is that it eliminates the need to integrate the constitutive equations iteratively. Two phases, the process phase and the solution phase, are introduced in the overall solution procedure. In the process phase, the stiffness matrix and initial stress vector are produced and then the displacement increments are solved globally. In the solution phase, the strain increments are substituted back and so the increments of stresses and state variables as well as other quantities can be solved locally. In one-dimensional cases, the procedure is relatively simple.

In this paper, an incremental formulation is proposed to analyze the buckling behavior of simply supported columns for various temperature and loading levels. The Bodner-Partom model is employed to represent the thermoelastoviscoplastic material behavior. As an example, B1900+Hf, a material that is used extensively in the gas turbine industry for high-temperture components, is presented. Results show that the model can predict both critical loads and postbuckling behavior under various temperature and load levels with the analysis not limited to long columns.

Incremental Formulation

Without derivation or proof, the equilibrium equation for an imperfect, simply supported column is given by

$$M_{,xx} - P(w_{,xx} + w^0_{,xx}) = 0 \tag{1}$$

where M is the moment, w the deflection, and w^0 the small initial geometric imperfection. Integrating the equation twice and using the boundary conditions to eliminate the constants, the equilibrium equation becomes

$$M - P(w + w^0) = 0 \tag{2}$$

Suppose that, at any instant t (position a), the moment, axial load, and the deflection are denoted by M_a, P_a, and w_a, respectively. Then, after a small time interval, the moment, load, and deflection are

$$M = M_a + \Delta M \tag{3a}$$

$$P = P_a + \Delta P \tag{3b}$$

$$w = w_a + \Delta w \tag{3c}$$

Substitution of these relations into Eq. (2) and use of equilibrium at position a yields

$$\Delta M - (P_a + \Delta P)\Delta w = \Delta P(w_a + w^0) \tag{4}$$

At any instant, the stress increment can be expressed as

$$\Delta\sigma = C\Delta\varepsilon - \Delta\zeta \tag{5}$$

where C and $\Delta\zeta$ are functions of temperature, deflection, and state variables at that instant. Using the relation

$$\Delta M = \int \Delta\sigma y \, dy, \qquad \Delta\varepsilon = -y\Delta w_{,xx}$$

and Eq. (5), the incremental equation, Eq. (4), becomes

$$\int Cy^2 \, dy \Delta w_{,xx} + (P_a + \Delta P)\Delta w$$
$$= -\Delta P(w_a + w^0) - \int \Delta\zeta y \, dy \tag{6}$$

Now let

$$\Delta w = \Delta w_c \sin\frac{n\pi x}{l}, \qquad w^0 = \Delta w_{ic} \sin\frac{n\pi x}{l}$$

which satisfies the boundary conditions, and assume

$$\Delta\zeta = \Delta\zeta_c \sin\frac{n\pi x}{l}$$

where the subscript c denotes the quantities at the middle of the column. Substituting these relations into Eq. (6) and letting $n = 1$, one can obtain the relation between the deflection increment and axial load increment,

$$\Delta w_c = \frac{\Delta P(w_{ac} + w_c^0) + \int \Delta\zeta_c y \, dy}{(\pi^2/l^2)\int Cy^2 \, dy - (P_a + \Delta P)} \tag{7}$$

where $P_a = \Sigma\Delta P$ and $w_{ac} = \Sigma\Delta w_c$.

The incremental formula can be written in terms of the displacement increment, so that the postbuckling behavior can be obtained. In this case, the formula becomes

$$\Delta P = \frac{\Delta w_c(\pi^2/l^2)\int Cy^2 \, dy - (P_a + \int \Delta\zeta_c y \, dy)}{w_{ac} + w_c^0 + \Delta w_c} \tag{8}$$

Some explanatory remarks are relevant at this point.

1) For linearly elastic material behavior, $\Delta\zeta = 0$ and $C = E$. Moreover, $\int Cy^2 dy = EI$. Then, Eq. (7) yields,

$$\Delta w_c = \frac{\Delta P(w_{ac} + w_c^0)}{P_{cr} - (P_a + \Delta P)} \tag{9}$$

where P_{cr} is the elastic Euler load. It is clear that the sum of the accumulated load and the incremental load tends to the critical load, the deflection increment becomes very large and eventually infinitely large.

2) In the limited viscoelastic (creep) case, the constitutive equation is

$$\dot\sigma + \frac{E_1 + E_2}{\eta}\sigma = E_1(\dot\varepsilon + \frac{E_2}{\eta}\varepsilon) \tag{10}$$

where the meaning of the parameters are shown in Fig. 1. If written in implicit incremental form, Eq. (10) becomes

$$\Delta\sigma = E_1\Delta\varepsilon - \frac{\Delta t E_2}{\eta}\left[\frac{(E_1 + E_2)\sigma}{E_2} - E_1\varepsilon\right] \tag{11}$$

Comparison of Eq. (11) with Eq. (5) yields

$$C = E_1, \qquad \Delta\zeta = \frac{\Delta t E_2}{\eta}\left[\frac{(E_1 + E_2)\sigma}{E_2} - E_1\varepsilon\right]$$

The critical creep load can be obtained by letting $\Delta P = 0$ in Eq. (7). In this case, if the term $\int \Delta\zeta y \, dy > 0$, there will be sequential deflection increments. Therefore, the criterion for a safe load is

$$\int \Delta\zeta y \, dy = 0$$

which leads to

$$\frac{E_1 + E_2}{E_2} P(w_a + w^0) + E_1 I w_{a,xx} = 0$$

or

$$P_{cr} = \left(\frac{E_1 + E_2}{E_2}\right)\left(\frac{w_{ac}}{w_{ac} + w_c^0}\right)\left(\frac{n^2\pi^2 E_1 I}{l^2}\right) = \left(\frac{E_1 + E_2}{E_2}\right)\left(\frac{w_{ac}}{w_{ac} + w_c^0}\right) P_{e1}$$

where $P_{e1} = (\pi^2 E_1 I)/l^2$ when n is taken to be 1. If we neglect the initial imperfection, the result will be the same with the conclusion given in Ref. 2.

3) In the elastoviscoplastic case, the Bodner-Partom constitutive equations for the one-dimensional case can be written as

$$\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^N, \qquad z = z^i + x \tag{12a}$$

$$\dot{\varepsilon}^N = f(\sigma,z) = \frac{2}{\sqrt{3}} \operatorname{sgn}(\sigma)\, D_0 \exp\left[-\frac{1}{2}\left(\frac{z}{\sigma}\right)^{2n}\right] \tag{12b}$$

$$\dot{x} = g(\sigma,\dot{\varepsilon}^N,x) = m_2(Z_3 - x)\sigma\dot{\varepsilon}^N - Z_1 Z_4 \left(\frac{x}{Z_1}\right)^{r_2} \tag{12c}$$

$$\dot{z}^i = h(\sigma,\dot{\varepsilon}^N,z^i) = m_1(Z_1 - z^i)\sigma\dot{\varepsilon}^N - Z_1 Z_5 \left(\frac{z^i - Z_2}{Z_1}\right)^{r_1} \tag{12d}$$

where $\dot{\varepsilon}^e$ and $\dot{\varepsilon}^N$ are the elastic and inelastic strain rates, respectively. There are two state variables, z^i and x, where z^i represents isotropic hardening and x represents directional hardening. In order to simplify the problem, isotropic hardening is neglected in the present calculations. In the constitutive equations, Eqs. (12), D_0, n, Z_1, Z_2, Z_3, Z_4, Z_5, r_1, r_2 are material constants. Some of them are temperature dependent.

Using Tanaka and Miller's method,[10] the increment of stress, strain, and state variables at any small time interval $\Delta t = t^b - t^a$ are expressed as

$$\Delta\sigma = \Delta t\{(1-\eta)\dot{\sigma}^a + \eta\dot{\sigma}^b\} \tag{13a}$$

$$\Delta\varepsilon = \Delta t\{(1-\eta)\dot{\varepsilon}^a + \eta\dot{\varepsilon}^b\} \tag{13b}$$

$$\Delta\varepsilon^N = \Delta t\{(1-\eta)\dot{\varepsilon}^{Na} + \eta\dot{\varepsilon}^{Nb}\} \tag{13c}$$

$$\Delta x = \Delta t\{(1-\eta)\dot{x}^a + \eta\dot{x}^b\} \tag{13d}$$

where η can be taken from 0 to 1. Using the Taylor expansion and retaining only linear terms, the constitutive equations become

$$\dot{\varepsilon}^{Nb} = f^a + f^a_{,\sigma}\Delta\sigma + f^a_{,x}\Delta x \tag{14a}$$

$$\dot{x}^b = g^a + g^a_{,\sigma}\Delta\sigma + g^a_{,\dot{\varepsilon}^N}\Delta\dot{\varepsilon}^N + g^a_{,x}\Delta x \tag{14b}$$

$$\dot{\sigma}^b = C(\dot{\varepsilon}^b - \dot{\varepsilon}^{Nb}) \tag{14c}$$

With the help of Eqs. (13), the quantities at the end of the time interval can be eliminated. Then, Eqs. (14) can be written as

$$\begin{bmatrix} -\eta\Delta t f^a_{,\sigma} & I & -\eta\Delta t f^a_{,x} & 0 \\ -\eta\Delta t g^a_{,\sigma} & -g^a_{,\dot{\varepsilon}^N} I & -\eta\Delta t g^a_{,x} & 0 \\ I & E & 0 & -E \end{bmatrix} \begin{Bmatrix} \Delta\sigma \\ \Delta\varepsilon^N \\ \Delta x \\ \Delta\varepsilon \end{Bmatrix} = \begin{Bmatrix} \Delta t[\dot{\varepsilon}^{Na} - \eta(\dot{\varepsilon}^{Na} - f^a)] \\ \Delta t[(\dot{x}^a - g_{,\dot{\varepsilon}^{Na}}\dot{\varepsilon}^{Na}) - \eta(\dot{x}^a - g^a)] \\ 0 \end{Bmatrix} \tag{15}$$

Equations (15) with four unknowns can be partially diagonalized, and then, the first three unknowns can be expressed in terms of $\Delta\varepsilon$,

$$\Delta\sigma = C\Delta\varepsilon - \Delta\zeta, \qquad \Delta x = -A_6\Delta\varepsilon + B_3$$

where

$$C = E\frac{1 - A_2A_6}{1 - A_1E}, \qquad \Delta\zeta = E\left(B_1 - \frac{A_2B_3}{1 - A_1E}\right)$$

$$A_1 = -\eta\Delta t \operatorname{sgn}(\sigma)\, D_0 \frac{2nz^{2n}}{\sqrt{3}\sigma^{2n+1}} \exp\left[-\frac{1}{2}\left(\frac{z}{\sigma}\right)^{2n}\right]$$

$$A_2 = \eta\Delta t \operatorname{sgn}(\sigma) \frac{2nz^{2n-1}}{\sqrt{3}\sigma^{2n}} \exp\left[-\frac{1}{2}\left(\frac{z}{\sigma}\right)^{2n}\right]$$

$$A_3 = -m_2(Z_3 - x)\sigma$$

$$A_4 = 1 + \eta\Delta t\left[m_2\sigma\dot{\varepsilon}^N + Z_4 r_2\left(\frac{x}{Z_1}\right)^{r_2-1}\right]$$

$$A_5 = -\eta\Delta t m_2(Z_3 - x)\dot{\varepsilon}^N$$

$$A_6 = \frac{A_5E(1 - A_1E) - A_1E(A_3 - A_5E)}{A_4(1 - A_1E) - A_2(A_3 - A_5E)}$$

$$B_1 = \Delta t[(1 - \eta)\dot{\varepsilon}^{Na} + \eta f^a]$$

$$B_2 = \Delta t[(\dot{x}^a + A_3\dot{\varepsilon}^{Na}) - \eta(\dot{x}^a - g^a)]$$

$$B_3 = \frac{B_2(1 - A_1E) - B_1(A_3 - A_5E)}{A_4(1 - A_1E) - A_2(A_3 - A_5E)}$$

Here, C and $\Delta\zeta$ are the quantities required in the incremental formulation and Δx is used in the calculation of the next step.

Numerical Example

A numerical example is given for the material B1900 + Hf.[8] The material constants are $m_1 = 0.27$ MPa^{-1}, $m_2 = 1.52$ MPa^{-1}, $Z_1 = 3000$ MPa, $Z_3 = 1150$ MPa, $r_1 = r_2 = 2$, and $D_0 = 10{,}000$ s^{-1}. The temperature-dependent constants are given in Table 1. For values in a temperature interval, linear interpolation is performed for their approximate estimation. The elastic modulus for the material is

$$E = 198{,}700 + 16.78T - 0.1034T^2 + 0.00001143T^3 \quad \text{MPa}$$

Figure 2 gives the critical load vs temperature for a column. The vertical coordinate for curve 1 is the critical load divided by the Euler load computed by using the value of the elastic modulus corresponding to the current temperature. The critical load before 800°C is nearly the same as and close to the Euler load. The critical load beyond 800°C decreases dramatically and the nondimensional load is 0.491 when the temperature reaches 1090°C. Curve 2 is the critical load divided by the Euler load at room temperature. The critical load continuously decreases as the temperature increases. In the first stage, the decrease is mainly caused by the decrease in

Table 1 Temperature-dependent constants for B1900 + Hf

	Temperature, °C			
Constants	$T \leqq 760$	871	982	1093
N	1.055	1.03	0.850	0.70
Z_0 (MPa)	2700	2400	1900	1200
Z_4 ($=Z_5$)(s^{-1})	0	0.0055	0.02	0.25
Z_2 ($=Z_0$)(MPa)	2700	2400	1900	1200

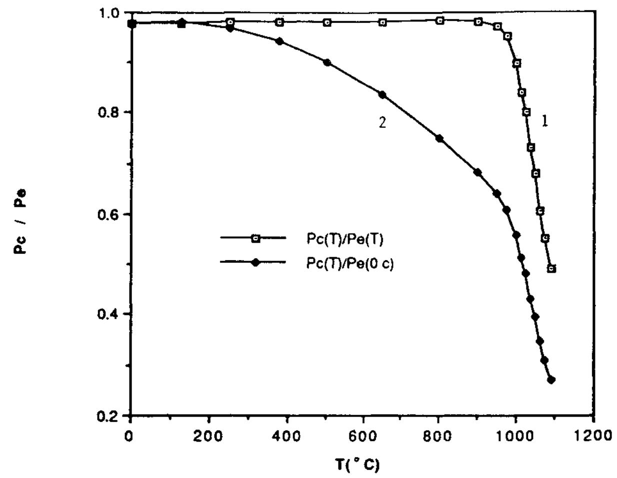

Fig. 2 Relation between critical load and temperature.

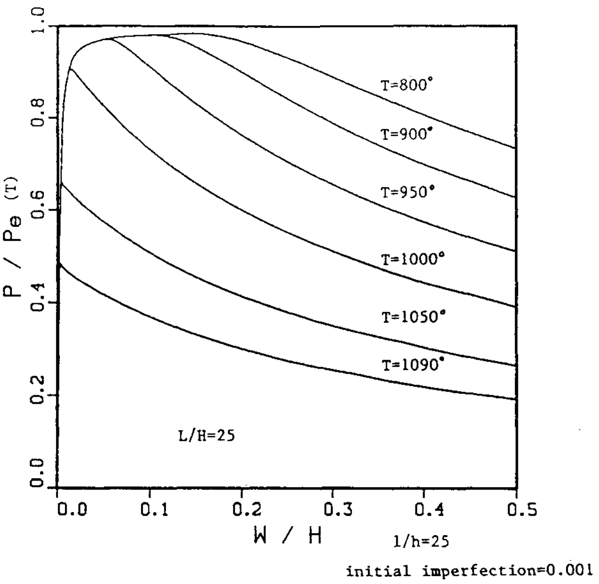

Fig. 3 Postbuckling curves.

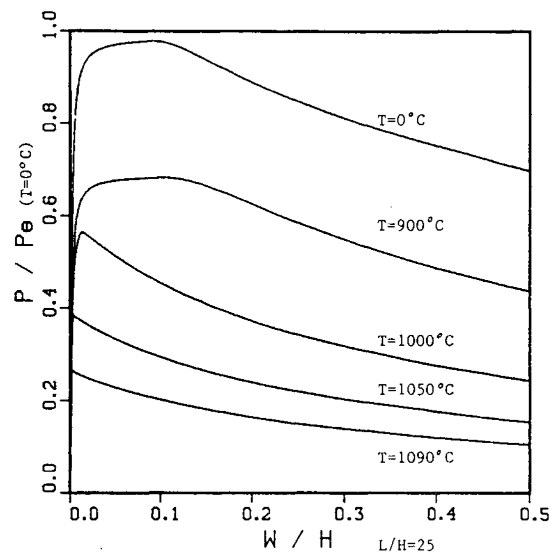

Fig. 4 Postbuckling curves under different temperatures.

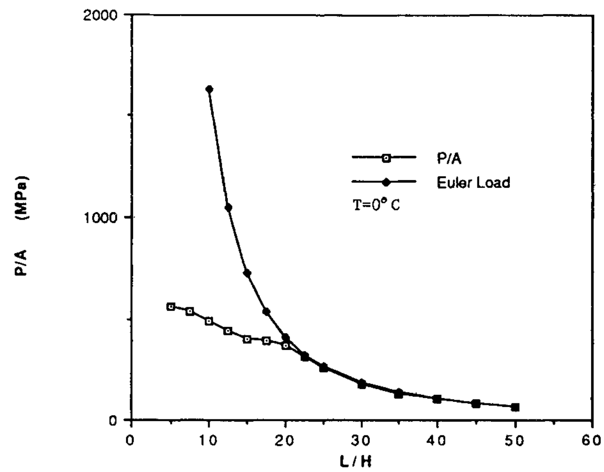

Fig. 5 Critical load vs slenderness ratio.

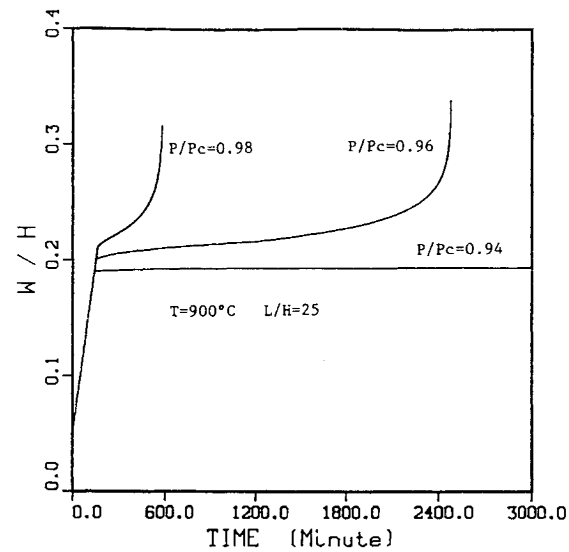

Fig. 6 Creep behavior for a column.

the value of the elastic modulus, whereas in the second stage, the decrease is caused by changes in other material parameters as well. Figure 3 shows the postbuckling behavior of the column. The horizontal coordinate is the nondimensional central deflection of the column, whereas the vertical coordinate is the nondimensionalized load corresponding to curve 1 in Fig. 2. Note that for $T < 1000°C$, the transition from the primary path to the secondary path is smooth (limit point on that curve). On the other hand, for $T \geq 1000°C$, the transition is sharp (cusp-like limit point). Small additional deflection from the equilibrium position could lead to a bigger decrease in the load carrying capability. Figure 4 gives the postbuckling curve in which the nondimensional critical load is plotted vs midpoint deflection. The nondimensionalized Euler load corresponds to room temperature. Figure 5 shows the critical load for columns of different slenderness ratios at room temperature. The curve coincides with the Euler buckling load curve at high slenderness ratios and gives correct load capability for the short columns (short column range). Figure 6 shows the creep behavior of a column. The temperature is kept at 900°C during the calculation. It takes 140 min to increase the load to the respected level and then keep the load constant. For the case in which the loading levels are 0.98 and 0.96, the critical times (to creep) are 570 and 2500 min, respectively. Therefore, in high-temperature cases, the critical time (to creep) for columns is very sensitive when the load level is close to the critical load.

Conclusions

An incremental formulation is given for the viscoplastic and viscoelastic postbuckling behavior of simply supported columns. The formulation has the advantage of presenting physical meaning and computational simplicity.

The nondimensional critical load for slender columns will decrease significantly when the temperature reaches a certain level. Below this value, the critical load is close to the Euler load.

The formulation works not only for slender columns but also for short and medium ranged columns. Therefore, no separation or special care is needed in calculations when different slenderness ratios are involved.

At high temperatures, the structure is very sensitive to small initial imperfections. The transition from the primary path to the secondary path becomes sharp; therefore, small external disturbance could then lead to the failure of the structure.

Acknowledgments

The work reported herein was supported by the NASA Lewis Research Center under Grant NAG3-534. The financial support provided by NASA is gratefully acknowledged. The authors also wish to thank Chris C. Chamis for his support, encouragement, and patience.

References

[1]Stubstad, J. M., and Simitses, G. J., "Solution Method for One-Dimensional Viscoelastic Problems," *AIAA Journal*, Vol. 26, No. 9, 1988, pp. 1127–1134.

[2]Dost, S., and Glockner, P. G., "On the Dynamic Stability of Viscoelastic Perfect Columns," *International Journal of Solids and Structures*, Vol. 18, No. 7, 1982, pp. 587–596.

[3]Szyszkowski, W., and Glockner, P. G., "The Imperfect Linearly Viscoelastic Column," *International Journal of Engineering Sciences*, Vol. 23, No. 10, 1985, pp. 1113–1120.

[4]Vinogradov, A. M., "Buckling of Viscoelastic Beam Columns," *AIAA Journal*, Vol. 25, No. 3, 1987, pp. 479–483.

[5]Stubstad, J. M., and Simitses, G. J., "Bounding Solutions of Geometrically Nonlinear Viscoelastic Problems," *AIAA Journal*, Vol. 24, No. 11, 1986, pp. 1843–1850.

[6]Perzyna, P., "Fundamental Problems in Viscoplasticity," *Advances in Applied Mechanics*, Vol. 9, Academic, New York, 1966.

[7]Miller, A. K., *Unified Constitutive Equations for Creep and Plasticity*, Elsevier, New York, 1987.

[8]Lindholm, U. S., Chan, K. S., Bodner, S. R., Weber, R. M., Walker, K. P., and Cassenti, B. N., "Constitutive Modeling for Isotropic Materials (Host)," NASA CR-174980, June 1985.

[9]James, G. H., Imbrie, P. K., Hill, P. S., Allen, D. H., and Haisler, W. E., "An Experimental Comparison of Several Current Viscoplastic Constitutive Models at Elevated Temperature," *ASME Journal of Engineering Materials and Technology*, Vol. 109, 1987, pp. 130–139.

[10]Tanaka, T. G., and Miller, A. K., "Development of a Method for Integrating Time-Dependent Constitutive Equations with Large, Small or Negative Strain Rate Sensitivity," *International Journal of Numerical Methods in Engineering*, Vol. 26, 1988, pp. 2457–2485.